ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 1

MATH00030

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Contents

1. Arithmetic and Algebra

1.1. Arithmetic of Numbers.

While we have calculators and computers to help us carry out calculations, it is also important to be able to perform these by hand. Calculators and computers are very good at working with numbers (provided we are careful to do the calculations in the correct order) but they are not nearly as good at working with algebraic expressions. Computers usually do not make mistakes working with algebraic expressions but you do not always end up with what you want. For example, you may ask the computer to simplify something but the expression it gives you can be more complicated than what you started with, or it may not be in the form you want. There is also the problem when working with numbers in that we may want an exact answer as a multiple of a root or as a multiple of π, say, but the calculator gives us an approximate decimal answer.

So we will start with numbers and then when we have found our feet we will progress to algebraic expressions. Before we do any calculations, we will have a look at the different sorts of numbers we will be working with throughout the course. Please don't worry too much about these at this stage, I have just included them for reference. The main thing is that you are able to do the calculations.

Different Types of Numbers

- Natural Numbers: These are the 'counting numbers' $1, 2, 3, \ldots$, where as is common in maths, I have used $\ldots$ to mean 'and so on'. Note that some books include 0 as a natural number, so it is important to check exactly what is meant in any given text. This is a general feature of Mathematics; notation varies from book to book, so please do bear this in mind. The set of all natural numbers is denoted by $\mathbb{N}$.
- Integers: These are the natural numbers together with their negatives and zero. That is $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$. Note that all natural numbers are also integers but not the other way around. The set of all integers is denoted by $\mathbb{Z}$.
- Rational Numbers: These are numbers of the form $\frac{a}{b}$ where both a and b are integers with b non-zero, such as $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{-3}{4} = -\frac{3}{4}$. Note that all integers are also rational numbers since, for example, we can write 2 as $\frac{2}{1}$. Also note that there are infinitely many different ways we can write each rational number. For example we can write $\frac{1}{2}$ as $\frac{2}{4}$ or $\frac{3}{6}$ etc. The set of all rational numbers is denoted by $\mathbb{Q}$.
- Real Numbers: These include all the rational numbers and also all the remaining numbers such as $\sqrt{2}$ (the number that when multiplied by itself gives 2) and π (the ratio of a circle's circumference to its diameter) that aren't rational (we call these irrational numbers). We all have an intuitive idea of what real numbers are and at this stage it is best to stick to that. There is a formal definition but it would take a whole third year course to explain it! The set of all real numbers is denoted by $\mathbb{R}$.
- Complex Numbers: These are numbers of the form $a + bi$ where a and b are real numbers and i is the 'number' with the property $i^2 = -1$. We will touch on these later in this course and there will also be a whole chapter on them in the mathematics course you will take in the second trimester. The set of all complex numbers is denoted by $\mathbb{C}$.

Now we have had a look at the different sorts of number we will be working with, let us do some calculations. We will start by looking at addition and subtraction. In fact these are really the same thing since subtraction can be regarded as the addition of a negative number. For example $3 - 2 = 3 + (-2) = 1$. Looked at like this, it doesn't matter which order we do the calculations in. For example, say we wanted to calculate $3 - 5 + 11$. Then a reasonable question is: Do we add 11 to $3-5$ or do we subtract $5+11$ from 3? However if we write the sum as $3+(-5)+11$ then it is clear that the answer must be 9. It doesn't matter if we add $3+(-5)$ to 11 or 3 to $(-5)+11$, we still end up with 9.

The question then arises, if we did want to subtract $5+11$ from 3, how would we write this down? The solution is to use brackets, these indicate that the sum inside the bracket should be calculated first. So $3-(5+11)$ is what we want.

The other main thing to remember when dealing with subtraction is that two minuses make a plus. So, for example, $2-(-3) = 2+3 = 5$. Note that here we write $-(-3)$ rather than $- - 3$.

Next let us have a look at how to add fractions and then we will do some examples.

Definition 1.1.1 (Addition of fractions)**.** Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Remark 1.1.2. If a, b, c and d are all integers then Definition 1.1.1 reduces to the concept of adding fractions by 'putting them over a common denominator' that you will have met in school. However Definition 1.1.1 is much more general than that. It could be that any of a, b, c and d are fractions or irrational numbers or algebraic expressions, for example.

Warning 1.1.3. Note that we need the conditions $b \neq 0$ and $d \neq 0$. This is because division by zero is not defined, so $\frac{1}{0}$, for example, does not mean anything and in particular it certainly isn't equal to zero!

Now let us do some examples. Later on in the chapter we will look at more general expressions, but for the moment we will stick to examples where a, b, c and d are integers.

Example 1.1.4.

(1) $\frac{2}{5} + \frac{1}{7} = \frac{(2)(7)+(5)(1)}{(5)(7)} = \frac{19}{35}$.

(2) $\frac{3}{4} - \frac{1}{7} = \frac{(3)(7)+(-1)(4)}{(4)(7)} = \frac{17}{28}$.

(3) $\frac{1}{2} - (-3) = \frac{1}{2} + \frac{3}{1} = \frac{(1)(1)+(3)(2)}{2} = \frac{7}{2}$.

(4) $\frac{1}{2} - \left(-\frac{1}{12}\right) = \frac{1}{2} + \frac{1}{12} = \frac{(1)(12)+(1)(2)}{(2)(12)} = \frac{14}{24} = \frac{7}{12}$.

Remark 1.1.5. Note that in the last example above, it would be also correct to use 12 as a common denominator instead of 24. We could also leave the answer as $\frac{14}{24}$, but if there is a common factor in the numerator and the denominator then it is more usual to cancel it.

Now that we have had a look at addition and subtraction, let us have a look at multiplication and division. As with addition and subtraction, these are also really opposite sides of the same coin. Division by a number a can be regarded as multiplication by the reciprocal of a, i.e., multiplication by $\frac{1}{a}$. The definitions are as follows.

Definition 1.1.6 (Multiplication of fractions)**.** Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Definition 1.1.7 (Division of fractions)**.** Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, where $b \neq 0$, $c \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

Remark 1.1.8. As with Definition 1.1.1, these definitions reduce to the definitions you will have met at school if a, b, c and d are all integers but are in fact much more general than that, since a, b, c and d could be fractions or irrational numbers or algebraic expressions.

Another important point to remember when multiplying or dividing is that two minuses make a plus. So multiplying two negative numbers (or algebraic expressions) will yield a positive number (or algebraic expression) and dividing a negative number by a negative number will yield a positive number and of course the same holds for algebraic expressions.

Now for some more examples, as above we will stick to the cases where a, b, c and d are integers for now but we will look at more complicated examples later on in the chapter.

Example 1.1.9.

(1) $\frac{1}{2} \times \frac{3}{4} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$.

(2) $-\frac{2}{3} \times \frac{5}{4} = \frac{(-2)(5)}{(3)(4)} = \frac{-10}{12} = -\frac{5}{6}$.

(3) $-2 \times \left(-\frac{7}{9}\right) = \frac{-2}{1} \times \frac{-7}{9} = \frac{(-2)(-7)}{(1)(9)} = \frac{14}{9}$.

(4) $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{(1)(4)}{(2)(1)} = \frac{4}{2} = 2$.

(5) $-\frac{5}{7} \div \frac{2}{9} = \frac{-5}{7} \times \frac{9}{2} = \frac{(-5)(9)}{(7)(2)} = -\frac{45}{14}$.

(6) $-\frac{1}{3} \div (-4) = \frac{-1}{3} \times \frac{-1}{4} = \frac{(-1)(-1)}{(3)(4)} = \frac{1}{12}$.

Remark 1.1.10. When there are minus signs involved, there are usually different ways to perform the calculation. For example in the last example, we could also use $-\frac{1}{3} \div (-4) = \frac{-1}{3} \times \frac{1}{-4} = \frac{(-1)(1)}{(3)(-4)} = \frac{-1}{-12} = \frac{1}{12}$. Of course, whichever approach we take, the final answer **MUST** be the same if our method is correct. Also note that if there is a negative number after a multiplication or division sign, then we always put the negative number in brackets.

For more complicated expressions, we also have to consider the order in which we should do the calculations. For example, does $3 + 4 \times 5$ mean that we calculate $3 + 4$ and multiply the result by 5 or does it mean multiply 4 by 5 and then add 3 to the result? The rule is that we first do any calculations in brackets, then any calculations involving multiplication and division (regarding division as multiplication by the reciprocal of the divisor) and finally any calculations involving addition and subtraction (regarding subtraction as addition of the negative of the original number). So $3 + 4 \times 5$ means $3 + (4 \times 5) = 3 + 20 = 23$. Note that in this case we perform the calculation on the right first, so it is important to realise that we don't always work from left to right. Here are some examples.

Example 1.1.11.

(1) $2 \div 3 \times 4 + 6 = 2 \times \frac{1}{3} \times 4 + 6 = \frac{2}{3} \times 4 + 6 = \frac{8}{3} + 6 = \frac{8+18}{3} = \frac{26}{3}$.

(2) $2 \div 3 \times (4 + 6) = 2 \div 3 \times 10 = 2 \times \frac{1}{3} \times 10 = \frac{2}{3} \times 10 = \frac{20}{3}$.

(3) $2 \div (3 \times 4 + 6) = 2 \div (12 + 6) = 2 \div 18 = \frac{2}{18} = \frac{1}{9}$.

(4) $2 \div (3 \times 4) + 6 = 2 \div 12 + 6 = \frac{2}{12} + 6 = \frac{1}{6} + 6 = \frac{1+36}{6} = \frac{37}{6}$.

We will do some more examples after we have covered indices: See Example 1.2.16.

1.2. **Powers, Roots, Rules of Indices and Order of Operations**.

In this section we will first look at powers and roots. In fact these are really the same thing as we shall see shortly. We will start with the simplest case.

Definition 1.2.1 (n'th power)**.** If n is a natural number and x is any number (even a complex number), then the *n'th power of x*, denoted x^n, is defined to be the product of x with itself n times.

Using this definition, we can now say what an n'th root is.

Definition 1.2.2 (n'th root)**.** If n is a natural number and x is a non-negative real number, then the *principal n'th root of x* is defined to be the non-negative number, denoted $\sqrt[n]{x}$, such that $\left(\sqrt[n]{x}\right)^n = x$.

Remark 1.2.3. If $n = 2$ (i.e., the square root) then we usually omit the 2 and just write $\sqrt{x}$. We also usually say 'cube root' rather than 'third root'. Definition 1.2.2 can be extended to the case where x is negative or indeed complex. We will do this in Mathematics 2 that you will take in the next trimester but for the moment we will stick to the case where x is non-negative. Also note that we sometimes write $x^{\frac{1}{n}}$ rather than $\sqrt[n]{x}$, so taking the n'th root of a number is the same thing as raising it to the $\frac{1}{n}$'th power.

Warning 1.2.4. Note that Definition 1.2.2 says '**non-negative** real number'. This is a point that causes a huge amount of confusion, so please do be careful. For example, the principal square root of 4 is 2. It is **NOT** -2 or ± 2. It is true that -2 is also a square root of 4 but this does **NOT** mean it is denoted by $\sqrt{4}$, it is denoted by $-\sqrt{4}$.

Before we go any further, let us do a few examples.

Example 1.2.5.

(1) $2^3 = 2 \times 2 \times 2 = 8$.
(2) $(-3)^2 = (-3) \times (-3) = 9$.
(3) $\left(\frac{1}{3}\right)^3 = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$.
(4) $\sqrt{9} = 9^{\frac{1}{2}} = 3$
(5) $\sqrt[3]{8} = 8^{\frac{1}{3}} = 2$
(6) $\sqrt[n]{0} = 0^{\frac{1}{n}} = 0$ for any natural number n.

Remark 1.2.6. There is a method for finding roots by hand but we won't cover it in this course. If we want to find roots to any particular degree of accuracy then computers or calculators can be used. Please do note however that in nearly all cases these are only approximations. For example, if we put $\sqrt{2}$ into a calculator, we will obtain something like 1.414213562. This is not the actual square root of 2 (which is a non-repeating non-terminating decimal) but only an approximation to it. So it would not be correct to write $\sqrt{2} = 1.414213562$. What we should write is $\sqrt{2} \simeq 1.414213562$ or $\sqrt{2} = 1.414213562$ (to 9 decimal places): See Section 1.4.

So far we have only defined powers where the power is either a natural number or the reciprocal of a natural number. The next definition covers the case where the power is a positive rational number.

Definition 1.2.7 (Positive rational powers)**.** If m and n are natural numbers and x is a non-negative real number, then the *$\frac{m}{n}$'th power of x*, denoted $x^{\frac{m}{n}}$ is defined to be $\left(\sqrt[n]{x}\right)^m = \left(x^{\frac{1}{n}}\right)^m$.

Remark 1.2.8. As was the case with Definition 1.2.2, this can be extended to the case where x is a negative real number or indeed a complex number and we will study this in the second trimester. It can also be generalised to the case where instead of $\frac{m}{n}$, we have a real or a complex number. This is the sort of problem that you would encounter in a third year undergraduate course, so we won't consider it in detail here. Note that the general approach for a real exponent is that we approximate it more and more closely by rational exponents and then 'take the limit'. We will look in more detail at limits when we study calculus.

We now know what x^a means if a is a positive real number, so the next we need to do is to extend the definition to the case when a is a negative real number. This is the next definition.

Definition 1.2.9 (Negative powers)**.** If a and x are positive real numbers, then the *$-a$'th power of x*, denoted x^{-a} is defined to be $\frac{1}{x^a}$.

Remark 1.2.10. As with the previous definitions, this can be extended to the case where x is negative or complex (but not zero since that would involve division by zero). Also note that the only power we have not yet defined is x^0. This will be dealt with in Theorem 1.2.12.

Now that we have made the definitions, let us do some examples.

Example 1.2.11.

(1) $27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2 = 3^2 = 9$.

(2) $16^{-\frac{3}{4}} = \dfrac{1}{16^{\frac{3}{4}}} = \dfrac{1}{\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.

(3) $\left(\dfrac{9}{4}\right)^{\frac{3}{2}} = \left(\sqrt[2]{\dfrac{9}{4}}\right)^3 = \left(\sqrt{\dfrac{9}{4}}\right)^3 = \left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$.

(4) $\left(\dfrac{81}{256}\right)^{-\frac{5}{4}} = \dfrac{1}{\left(\dfrac{81}{256}\right)^{\frac{5}{4}}} = \dfrac{1}{\left(\sqrt[4]{\dfrac{81}{256}}\right)^5} = \dfrac{1}{\left(\dfrac{3}{4}\right)^5} = \dfrac{1}{243/1024} = \dfrac{1024}{243}$.

We will also need to have some general rules that will allow us to manipulate indices and these are listed in the following theorem.

Theorem 1.2.12 (Rules of indices)**.** *Let x be a positive real number and let m and n be real numbers, then*

(1) $x^m \times x^n = x^{m+n}$.
(2) $(x^m)^n = x^{mn}$.
(3) $x^m \div x^n = x^{m-n}$.
(4) $x^0 = 1$.
(5) $x^1 = x$.

Remark 1.2.13. In fact most of these rules hold when x is zero or negative but there are some exceptions and sometimes if x is negative then we will end up with complex numbers, so we do need to be careful. In this course I will not ask you about any of these tricky situations however.

I think it is also important to understand where these results have come from, since if you do, you don't have to memorize them, you can just derive them whenever you need them. So below I will indicate how the results can be justified if m and n are natural numbers. If you find the arguments hard to follow, then my advice is to first use actual numbers for m and n. You will then be better able to understand the general case.

(1) x^m means x multiplied by itself m times and x^n means x multiplied by itself n times, so $x^m \times x^n$ is x multiplied by itself $m + n$ times. That is x^{m+n}. Expressed algebraically

$$x^m \times x^n = \underbrace{x \times \cdots \times x}_{m \text{ terms}} \times \underbrace{x \times \cdots \times x}_{n \text{ terms}} = \underbrace{x \times \cdots \times x}_{m+n \text{ terms}} = x^{m+n}.$$

Note that here we have used the notation '$\cdots$' which means 'and so on'.

(2) x^m means x multiplied by itself m times and we are multiplying this by itself n times, so we end up with x multiplied by itself mn times, which is x^{mn}. Expressed algebraically

$$(x^m)^n = \underbrace{x^m \times \cdots \times x^m}_{n \text{ terms}} = \underbrace{x \times \cdots \times x}_{mn \text{ terms}} = x^{mn}.$$

(3) Here we will first look at the case when $m > n$. $x^m \div x^n$ means we start with x multiplied by itself m times but we divide by x multiplied by itself n times, so these cancel with n of the x's in the numerator and we are left with x multiplied by itself $m - n$ times, which is x^{m-n}. Expressed algebraically

$$x^m \div x^n = \frac{\underbrace{x \times \cdots \times x}_{m \text{ terms}}}{\underbrace{x \times \cdots \times x}_{n \text{ terms}}} = \underbrace{x \times \cdots \times x}_{m-n \text{ terms}} = x^{m-n}$$

Now suppose that $m < n$. Then, using the above equation,

$$x^m \div x^n = \frac{1}{x^n \div x^m} = \frac{1}{x^{n-m}} = x^{-(n-m)} = x^{m-n}.$$

(4) This is Case (3) with $m = n$. We will assume (3) and see where this leads us. On the left hand side of $x^m \div x^n = x^{m-n}$ we have $x^m \div x^m$ which is 1 since anything (except 0) divided by itself is 1. The right hand side is $x^{m-m} = x^0$, so we obtain $x^0 = 1$. Note this is not true if $x = 0$ since 0^0 is not defined, i.e., it has no meaning, so it can't be equal to anything.

(5) This is case (3) with $m = n + 1$. For example if we take $n = 2$ and $m = 3$ the left hand side of $x^m \div x^n = x^{m-n}$ is $x^3 \div x^2 = \dfrac{x \times x \times x}{x \times x} = x$ while the right hand side is $x^{3-2} = x^1$ and so we obtain $x^1 = x$.

Warning 1.2.14. The above justifications are not rigorous and would not be considered proofs. For example we didn't consider the case where m and n are not integers and in (3) we assumed the case when $m = n$. It would be really quite hard to give a rigorous proof of the general case where m and n are real numbers, but I still think the justifications are very useful, since it gives you an idea of where the rules come from and this makes them much easier to understand, remember and use.

Although the above will hopefully help you remember the rules, the most important thing is to be able to use them, so here are some examples.

Example 1.2.15.

(1) $x^{10} \times x^{12} = x^{10+12} = x^{22}$.

(2) $x^9 \times x^{-11} = x^{9+(-11)} = x^{-2} = \dfrac{1}{x^2}$.

Note that x^{-2} would also be acceptable as a final answer here.

(3) $x^{\frac{1}{2}} \times x^{\frac{2}{3}} = x^{\frac{1}{2}+\frac{2}{3}} = x^{\frac{3+4}{6}} = x^{\frac{7}{6}}$.

(4) $(x^2)^3 = x^{2\times 3} = x^6$.

(5) $(x^{-2})^{-3} = x^{-2\times(-3)} = x^6$.

(6) $\left(x^{\frac{2}{3}}\right)^{-\frac{5}{4}} = x^{\frac{2}{3}\times(-\frac{5}{4})} = x^{-\frac{10}{12}} = x^{-\frac{5}{6}}$.

In this case $\dfrac{1}{x^{\frac{5}{6}}}$ would also be acceptable as an answer.

(7) $x^3 \div x^7 = x^{3-7} = x^{-4} = \dfrac{1}{x^4}$.

Here we could also leave the answer as x^{-4}.

(8) $x^{-\frac{1}{2}} \div x^{-\frac{2}{3}} = x^{-\frac{1}{2}-\left(-\frac{2}{3}\right)} = x^{-\frac{1}{2}+\frac{2}{3}} = x^{\frac{1}{6}}$.

(9) $1^0 = 1$.

(10) $\pi^0 = 1$.

(11) $\left(x^{\frac{4}{5}} \times x^{-\frac{2}{3}}\right)^{-\frac{3}{4}} = \left(x^{\frac{4}{5}-\frac{2}{3}}\right)^{-\frac{3}{4}} = \left(x^{\frac{2}{15}}\right)^{-\frac{3}{4}} = x^{\frac{2}{15}\times\left(-\frac{3}{4}\right)} = x^{-\frac{6}{60}} = x^{-\frac{1}{10}} = \dfrac{1}{x^{\frac{1}{10}}}$.

Here we could also leave the answer as $x^{-\frac{1}{10}}$.

As we did with multiplication/division and addition/subtraction, we also have to consider which operation do we perform first. The order is as follows: Brackets have the highest priority, followed by powers/roots (which as we saw above are essentially the same thing), then multiplication/division and finally addition/subtraction. There are several different mnemonics for remembering this order: BIMDAS is one. This stands for Brackets Indices Multiplication Division Addition Subtraction. Do also remember however that multiplication and division have equal priority as do addition and subtraction. Here are some examples.

Example 1.2.16.

(1) $2 \times 3^2 = 2 \times 9 = 18$.

(2) $(2 \times 3)^2 = 6^2 = 36$.

(3) $2 \div 3^3 + 2 = 2 \div 27 + 2 = 2 \times \dfrac{1}{27} + 2 = \dfrac{2}{27} + 2 = \dfrac{2+54}{27} = \dfrac{56}{27}$.

(4) $2 \div (3^3 + 2) = 2 \div (27 + 2) = 2 \div 29 = \dfrac{2}{29}$.

(5) $(2 \div 3)^3 + 2 = \left(\dfrac{2}{3}\right)^3 + 2 = \dfrac{2^3}{3^3} + 2 = \dfrac{8}{27} + 2 = \dfrac{8+54}{27} = \dfrac{62}{27}$.

(6) $2 \div 3 \times 4 + 5^2 = 2 \div 3 \times 4 + 25 = 2 \times \dfrac{1}{3} \times 4 + 25 = \dfrac{2}{3} \times 4 + 25 = \dfrac{8}{3} + 25 = \dfrac{8+75}{3}$
$= \dfrac{83}{3}$.

(7) $2 \div 3 \times (4+5)^2 = 2 \div 3 \times 9^2 = 2 \div 3 \times 81 = 2 \times \dfrac{1}{3} \times 81 = \dfrac{2}{3} \times 81 = \dfrac{162}{3} = 54$.

(8) $2 \div (3 \times 4 + 5)^2 = 2 \div (12+5)^2 = 2 \div 17^2 = 2 \div 289 = \dfrac{2}{289}$.

(9) $2 \div (3 \times 4 + 5^2) = 2 \div (3 \times 4 + 25) = 2 \div (12 + 25) = 2 \div 37 = \dfrac{2}{37}$.

(10) $(2 \div 3 \times 4 + 5)^2 = \left(2 \times \dfrac{1}{3} \times 4 + 5\right)^2 = \left(\dfrac{2}{3} \times 4 + 5\right)^2 = \left(\dfrac{8}{3} + 5\right)^2$
$= \left(\dfrac{8+15}{3}\right)^2 = \left(\dfrac{23}{3}\right)^2 = \dfrac{23^2}{3^2} = \dfrac{529}{9}$.

(11) $(2\div 3\times(4+5))^2 = (2\div 3\times 9)^2 = \left(2\times\dfrac{1}{3}\times 9\right)^2 = \left(\dfrac{2}{3}\times 9\right)^2 = \left(\dfrac{18}{3}\right)^2 = 6^2$
$= 36.$

Warning 1.2.17. There still remains the question of what x^{y^z} means. Does it mean $(x^y)^z$ or $x^{(y^z)}$? It would be expected that this should mean $(x^y)^z$ since that is what we get if we work from left to right. However it is in fact taken to mean $x^{(y^z)}$. This is a bit strange and the explanation is that if we wanted to write $(x^y)^z$ we would write $x^{y\times z}$, so we take x^{y^z} to mean $x^{(y^z)}$. What I suggest in situations like this is to use brackets to make it absolutely plain what you mean. In fact this is a general piece of advice; if you think a particular expression could be ambiguous then it is better to use extra brackets to explain what you mean. You can never go wrong by using extra brackets but sometimes you can by not using enough.

Theorem 1.2.12 dealt with the case where we were only dealing with powers of a single number or variable. However sometimes we have a power of a product and in this case the following theorem can prove to be very useful.

Theorem 1.2.18. *Le x and y be positive real numbers and let a be a real number, then*

$$(x\times y)^a = x^a \times y^a. \tag{1}$$

Remark 1.2.19. As was the case with Theorem 1.2.12, this theorem is also true in a lot of cases when x and y are not positive but these cases can prove to be complicated and can involve complex numbers, so I won't deal with these cases here.

Also note that when writing mathematics, the multiplication sign $\times$ is often replaced with a dot or indeed omitted altogether. So (1) can also be written $(x\cdot y)^a = x^a\cdot y^a$ or even $(xy)^a = x^a y^a$. From now on I will use any of these notations interchangeably; one thing you have to get used to in maths is the wide variety of notation used, even when exactly the same thing is meant. Of course we do also have to guard against ambiguity. For example, if we want to denote 2 times 3 then we have to be careful writing $2\cdot 3$ since this could be mistaken for the decimal 2 point 3 and of course we can't omit the dot altogether since then we would have twenty three.

Now let us have a look at some examples which use Theorem 1.2.18.

Example 1.2.20.

(1) $(\sqrt{x}y)^2 = (\sqrt{x})^2 y^2 = xy^2.$

(2) $(x^{-2}y^3)^{-4} = (x^{-2})^{-4}(y^3)^{-4} = x^{-2(-4)}y^{3(-4)} = x^8y^{-12} = \dfrac{x^8}{y^{12}}.$
Here we could also leave the answer as x^8y^{-12}.

(3) $(x^2y^3z^4)^5 = ((x^2y^3)z^4)^5 = (x^2y^3)^5(z^4)^5 = (x^2)^5(y^3)^5(z^4)^5 = x^{2\times 5}y^{3\times 5}z^{4\times 5}$
$= x^{10}y^{15}z^{20}.$

1.3. **Logarithms** .

In Section 1.2 we saw that roots and indices are the same thing and in this section we will see that logarithms are also indices. Let us start with the definition.

Definition 1.3.1 (Logarithm)**.** Let a and x be real numbers with $a > 1$ and $x > 0$. Then *the logarithm of x to the base a*, denoted $\log_a x$, is the number y such that $x = a^y$.

Remark 1.3.2. That is the logarithm is the power a has to be risen to in order to obtain x. That is why I said above that logarithms are indices.

Sometimes logarithms are also defined if $0 < a < 1$ but these are so rarely used I won't deal with them here (if I did, it would cause complications below).

Warning 1.3.3. Note that x has to be a positive number. The log of zero or a negative number is not defined, i.e., it doesn't exist.

While we will usually use a calculator to find logs, I think it is also important to start out by doing some simple ones by hand, since this will help you to understand the concept. So here are some examples.

Example 1.3.4.

(1) Find $\log_2 8$.
Since $2^3 = 8$, it follows that $\log_2 8 = 3$.

(2) Find $\log_3 3$.
Since $3^1 = 3$, it follows that $\log_3 3 = 1$.
In fact $\log_a a = 1$ for any $a > 0$, since $a^1 = a$.

(3) Find $\log_4 2$.
Since $4^{\frac{1}{2}} = 2$, it follows that $\log_4 2 = \frac{1}{2}$.

(4) Find $\log_{10} 1$.
Since $10^0 = 1$, it follows that $\log_{10} 1 = 0$.
In fact $\log_a 1 = 0$ for any $a > 0$, since $a^0 = 1$.

(5) Find $\log_5 \left(\dfrac{1}{25}\right)$.
Since $\dfrac{1}{25} = 5^{-2}$, it follows that $\log_5 \left(\dfrac{1}{25}\right) = -2$.

(6) Find $\log_8 \left(\dfrac{1}{2}\right)$.
Since $\dfrac{1}{2} = 8^{-\frac{1}{3}}$, it follows that $\log_8 \left(\dfrac{1}{2}\right) = -\frac{1}{3}$.

Note that in the above examples, $\log_a x > 0$ if $x > 1$ and $\log_a x < 0$ if $x < 1$. In fact this is always true.

Proposition 1.3.5. *Let a and x be real numbers with $a > 1$ and $x > 0$. Then*

(1) $\log_a x < 0$ *if* $x < 1$.
(2) $\log_a 1 = 0$.
(3) $\log_a x > 0$ *if* $x > 1$.

As a ‘challenge problem’, see if you can figure out why (1) and (3) hold.

I think it is also useful to see what the graph of the logarithm function looks like (we will return to graphs in Chapter 4, where I will formally say what they are). In Figure 1 I have plotted the graph for various values of a. This figure gives a good visual illustration of Proposition 1.3.5. Note that the graphs all lie to the right of the y-axis (since log is only defined for $x > 0$), they all lie below the x-axis for $0 < x < 1$ (this is (1)), they all cut the x-axis at $x = 1$ (this is (2)) and they all lie above the x-axis for $x > 1$ (this is (3)).

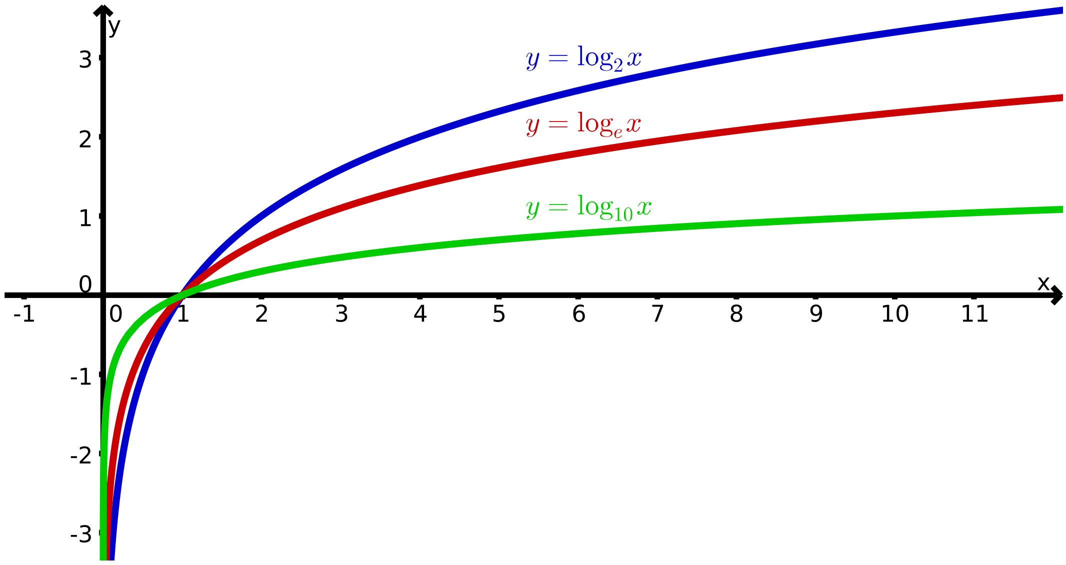

FIGURE 1. The graphs of $y = \log_a x$ for $a = 2$, $a = e$ and $a = 10$.

I have just plotted the graphs for $a = 2$, $a = e$ and $a = 10$, since these are the values of a that you will most likely meet. Logs to the base 2 are used in Computer Science. Logs to the base 10 were often used for calculation (we will see in Theorem 1.3.7, items (1) and (2) that logs convert multiplication into addition and taking powers to multiplication) in the days before calculators but are not so common these days. Logs to the base e arise naturally in many different areas of science and this will most probably be the base that you encounter the most. We will meet e again when we study calculus, but for the moment we will note that it is a number as important in calculus as π is important in geometry and trigonometry. Like π, e is a special sort of irrational number called a transcendental number. You don’t need to know what this means for this course, but for those of you that have met polynomials, it means that it is not the root of a polynomial with integer coefficients. In common with all irrational numbers, e cannot be written down exactly as a decimal, since it doesn’t repeat or terminate. It is approximately equal to 2.718 to three decimal places (we will look at decimal places in Section 1.4).

Please also note some other features of all the functions. The all get very large and negative as x gets close to zero, they all increase as x increases and they all get very large as x gets very large (this last feature may not be completely apparent from the graph but it is true).

Warning 1.3.6. We also have to be careful with notation when dealing with logs since different books use different notation. Depending on the book (or course) $\log x$ (where the subscript is omitted) may mean log to the base e or log to the base 10, so it is essential at the start to find out exactly what it does mean. Logs to the base e may also be denoted by $\ln x$ and this is what I would recommend if you have the choice since $\ln x$ never stands for anything else.

As was the case with indices, there are several rules that allow us to simplify logarithms. These, together with some points covered above, are listed in the next theorem.

Theorem 1.3.7 (Rules of logarithms)**.** *Let a, b, m, x and y be real numbers with $a, b > 1$ and $x, y > 0$. Then*

(1) $\log_a(xy) = \log_a x + \log_a y$.
(2) $\log_a (x^m) = m \log_a x$.
(3) $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.
(4) $\log_a 1 = 0$.
(5) $\log_a a = 1$.
(6) $\log_a x = \dfrac{\log_b x}{\log_b a}$.
(7) $a^{\log_a x} = x$.
(8) $\log_a (a^x) = x$.

Remark 1.3.8. In fact the first five items in Theorem 1.3.7 follow from the corresponding items in Theorem 1.2.12. I won't give the details here, since at this stage it is more important to be able to use the rules rather than to prove them. However I think just knowing that they follow will make it easier to get used to both sets of rules. If you want to try a 'challenge problem', then have a go at trying to prove the first three items.

Item (6) enables us to calculate a log to any base we want even though most calculators are only able to calculate logs to the bases e and 10.

Item (7) is just the definition of a log and (8) follows from (2) and (5).

Warning 1.3.9. Note that in general it is **NOT** true that $\log_a(x+y) = \log_a x + \log_a y$ or that $\log_a(xy) = \log_a x \times \log_a y$.

Now let us do some examples where we use Theorem 1.3.7 to simplify various expressions.

Example 1.3.10.

(1) $\log_a (xy^2) = \log_a x + \log_a (y^2) = \log_a x + 2\log_a y$.
(2) $\log_a \left(\left(\dfrac{x}{y}\right)^4\right) = 4\log_a \left(\dfrac{x}{y}\right) = 4(\log_a x - \log_a y) = 4\log_a x - 4\log_a y$.

(3) $\log_a \left(\dfrac{x^3}{y^{\frac{1}{2}}}\right) = \log_a (x^3) - \log_a \left(y^{\frac{1}{2}}\right) = 3\log_a x - \dfrac{1}{2}\log_a y.$

(4) $\log_a \left(x^{\log_b a}\right) = \log_b a \times \log_a x = \log_b x.$

1.4. **Decimal Places, Significant Figures and Scientific Notation**.

While in pure mathematics, we often deal with algebraic expressions or exact values, when we use mathematics in other scientific disciplines we will usually be dealing with data which will be approximate. In this section, we will start to look at the way approximations should be treated.

Firstly, I think the most important thing to realise is that $=$ has a very specific meaning in mathematics. It means that two expressions or numbers are equal and it should only be used in this precise situation. It should **NOT** be used (at least on its own) if two expressions or numbers are only approximately equal. Note that even if two numbers agree to the first billion digits, this does not mean that they are equal. Also note that if you calculate something using a calculator, this does not mean that the answer is 'correct'. For example, if I put $\dfrac{\pi}{2}$ into my calculator I get 1.570796327. This does not mean that $\dfrac{\pi}{2} = 1.570796327$. In fact $\dfrac{\pi}{2} = 1.570796327$ can't be true since $\dfrac{\pi}{2}$ is a non-repeating, non-terminating decimal.

Now that we have had a look at what we should not do, we will look at what we should do. First let us look at rounding positive numbers to a given number of decimal places. Say we want to round a positive number to n decimal places. What we do is to look at the $n+1$'th decimal place and

- If it is five or more we round up.
- If it is four or less we round down.

Warning 1.4.1. Note that there can be cases where more than the n'th decimal place changes. For example 19.96 is 20.0 to one decimal place, so that in this case all three of the digits we end up with are different to those we start with.

Also note that if we give an answer to n decimal places, we should always write down n decimal places, even if there are zeros on the right (these are sometimes called trailing zeros). So, in the above example 20 would not be correct, it must be 20.0 if we are giving the answer to one decimal place.

Here are some examples.

Example 1.4.2.

(1) $21.543 = 21.5$ to one decimal place.
Note that here we are allowed to use '=' since we indicate that we are only saying the numbers are equal to one decimal place.
Also note that 'one decimal place' is often abbreviated to '1 d.p.'

(2) $35.235 = 35.24$ to 2 d.p.

(3) $0.000423 = 0.000$ to 3 d.p.

(4) $5 = 5.00$ to 2 d.p.
Note that although this is correct mathematically (where we just quote the

number to the given number of decimal places), it might not be correct in physics, for example, since the 5 may just mean that the measurement is to the nearest integer.

Now let us look at rounding negative numbers to n decimal places. This is another area where different books can tell you different things but if we want to be consistent with the way we rounded positive numbers then the rules have to be as follows: We look at the $n+1$'th decimal place and

- If it is five or less we round up.
- If it is six or more we round down.

This may seem a bit strange but if we study the number line in Figure 2, hopefully everything will become clear.

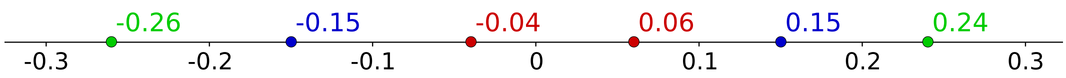

FIGURE 2. Number line illustrating rounding to one decimal place.

The green points (-0.26 and 0.24) both lie closer to the black numbers on the left (-0.3 and 0.2) so both should be rounded down (that is to the left). So $-0.26 = -0.3$ and $0.24 = 0.2$ both to 1 d.p. The blue points (-0.15 and 0.15) both lie exactly between the black numbers to the left and the right (-0.2 and -0.1, and 0.1 and 0.2, respectively) so, as a rule of thumb, we are going to round them up (that is to the right). So $-0.15 = -0.1$ and $0.15 = 0.2$ both to 1 d.p. The red points (-0.04 and 0.06) both lie closer to the black numbers on the right (0 and 0.1) so both should be rounded up (that is to the right). So $-0.04 = 0.0$ and $0.06 = 0.1$ both to 1 d.p.

Here are some more examples involving negative numbers.

Example 1.4.3.

(1) $-1.66 = -1.7$ to 1 d.p.
(2) $-5.2755 = -5.275$ to 3 d.p.
(3) $-0.455 = -0.45$ to 2 d.p.
(4) $-3.999 = -4.00$ to 2 d.p.

While approximating a number to a particular number of decimal places is sometimes a good idea, there will be other situations where it is not appropriate. For example if we are measuring some astronomical distance and we can measure it to the nearest 1% then it would not be possible to give the measurement in miles to any number of decimal places, since we don't have sufficient accuracy. In situations like this it is more appropriate to give our answer to a certain number of significant figures. The first significant figure is the first non-zero digit counting from the left, and the second significant figure is the next digit to the right of this, and so on. So, in the above situation, where we can measure something to the nearest 1%, we could quote the measurement to two significant figures.

The actual rules for rounding are the same as those we used for decimal places, so let us look at some examples.

Example 1.4.4.

(1) $64528562 = 65000000$ to two significant figures.
Note that 'two significant figures' is often abbreviated to '2 s.f.'
(2) $0.003456 = 0.00346$ to 3 s.f.
(3) $0.999 = 1$ to 1 s.f.
(4) $6 = 6.000$ to 4 s.f.
Note again here that while this is correct mathematically, it might not be appropriate in a given scientific situation.
(5) $-2.886 = -2.89$ to 3 s.f.
(6) $-0.5 = -0.50$ to 2 s.f.
(7) $-0.6 = -0.6$ to 1 s.f.

As we noted above, 65000000 to 2 s.f. might be a reasonable way to give our measurement if it is accurate to 1% but this is still not really the best way to present our data, since someone else reading our report has to go to the trouble of counting the number of zeros. This is not too much work in this case but if there were 35 zeros, say, it would entail quite a lot of effort. This is where scientific notation comes into its own. The idea is to write a number as $x \times 10^a$, where a is an integer and is usually chosen so that $1 \leqslant x < 10$ for positive numbers and $-10 < x \leqslant -1$ for negative numbers (note these two conditions can also be written $1 \leqslant |x| < 10$, where $|x|$ is the absolute value of x). Here are some examples (the numbers on the right are in scientific notation).

Example 1.4.5.

(1) $2395738 = 2.395738 \times 10^6$.
(2) $0.0000635 = 6.35 \times 10^{-5}$.
(3) $5.53 = 5.53 \times 10^0$.
Note that sometimes we omit the 10^0 in cases like this.
(4) $34 = 3.4 \times 10^1$.
(5) $0.9 = 9 \times 10^{-1}$.

Remark 1.4.6. The examples above show that to convert any number to scientific notation, we count the number of places we have to move the decimal point (counting left as positive and right as negative) so that it ends up to the right of the first non-zero digit (from the left) and this number then becomes the exponent of the 10.

Of course, sometimes we will want to convert a number in scientific notation back to ordinary notation and in this case we just reverse the above process. That is we move the decimal point by the number of places indicated by the exponent of 10 (where a negative number means move the decimal point to the left and a positive number means move it to the right). Here are some examples.

Example 1.4.7.

(1) $2.653 \times 10^7 = 26530000$.
(2) $4.21 \times 10^{-5} = 0.0000421$.
(3) $3.21 \times 10^{25} = 32100000000000000000000000$.
This last example shows why scientific notation is so useful.

We can also combine scientific notation with significant figures (but not usually with decimal places) and this is quite often the way data is displayed in science. Here are some examples.

Example 1.4.8.

(1) $73857647 = 7.4 \times 10^7$ to 2 s.f.
(2) $0.00004555 = 4.56 \times 10^{-5}$ to 3 s.f.
(3) $-2345.5 = -2.345 \times 10^3$ to 4 s.f.
(4) $-0.0005 = -5.0000 \times 10^{-4}$ to 5 s.f.
Note again that while this is correct mathematically, it might not be appropriate scientifically.

1.5. **Arithmetic of Algebraic Expressions.**

Although we have already manipulated some algebraic expressions, we are now going to look in more detail at adding, subtracting, multiplying and dividing algebraic expressions. In this chapter we will deal with expressions containing powers of x.

1.5.1. *Addition of Algebraic Expressions*.

As we did when we dealt with numbers in Section 1.1 we will deal with addition and subtraction at the same time since they are essentially the same thing. The main idea is to collect together terms that have the same power and then add their coefficients. As is usual with maths the best way to learn is to have a look at some examples and then do some problems, so here are a few examples.

Example 1.5.1.

(1)
$$\begin{aligned}(x^2+3x-2)+(x^3-2x^2+5x+2)\\ &= x^3+(x^2-2x^2)+(3x+5x)+(-2+2)\\ &= x^3-x^2+8x.\end{aligned}$$

(2)
$$\begin{aligned}(-x^4+5x^3-3x+5)-(6x^3-4x^2-3)\\ &= -x^4+(5x^3-6x^3)-(-4x^2)-3x+(5-(-3))\\ &= -x^4-x^3+4x^2-3x+8.\end{aligned}$$

(3)
$$\begin{aligned}(x^5-6x^2+2-2x^{-2})+(-x^5-4x^3-4x^2-x^{-1}-3x^{-2})\\ &= (x^5-x^5)-4x^3+(-6x^2-4x^2)+2-x^{-1}+(-2x^{-2}-3x^{-2})\\ &= -4x^3-10x^2+2-x^{-1}-5x^{-2}.\end{aligned}$$

Remark 1.5.2. It is not necessary to include the middle step in each of these calculations once you have got the hand of things. I have just included it to make it obvious exactly what is happening.

1.5.2. *Multiplication of Algebraic Expressions*.

Next we come to multiplication. While not difficult, we do have to keep our wits about us here, since it is easy to make a mistake if we are not careful. The main thing to remember is that if we are multiplying two expressions together, then each term in the first expression has to be multiplied by each term in the second expression. These individual terms are multiplied by multiplying the coefficient and multiplying the powers of x using the rules of indices. Here are some examples to show you how it works.

Example 1.5.3.

(1)

$$\begin{aligned} 2x^2(3x+4x^2) &= (2x^2)(3x)+(2x^2)(4x^2) \\ &= (2)(3)x^{2+1}+(2)(4)x^{2+2} \\ &= 6x^3+8x^4. \end{aligned}$$

(2)

$$\begin{aligned} (3x^3-5)(2x^2+3x) &= 3x^3(2x^2+3x)-5(2x^2+3x) \\ &= (3x^3)(2x^2)+(3x^3)(3x)+(-5)(2x^2)+(-5)(3x) \\ &= (3)(2)x^{3+2}+(3)(3)x^{3+1}+(-5)(2)x^2+(-5)(3)x \\ &= 6x^5+9x^4-10x^2-15x. \end{aligned}$$

(3)

$$\begin{aligned} (2x^2+1)^2 &= (2x^2+1)(2x^2+1) \\ &= 2x^2(2x^2+1)+1(2x^2+1) \\ &= (2x^2)(2x^2)+(2x^2)(1)+(1)(2x^2)+(1)(1) \\ &= (2)(2)x^{2+2}+2x^2+2x^2+1 \\ &= 4x^4+4x^2+1. \end{aligned}$$

Note that this example shows that we can also calculate powers of an algebraic expression using this technique, provided the power is a natural number. We will return to the subject of finding powers of algebraic expressions when we study The Binomial Theorem in Section 1.6.

(4)

$$
\begin{aligned}
(-3x^2+5x-3)(2x^2-3x^{-1})& \\
&= -3x^2(2x^2-3x^{-1})+5x(2x^2-3x^{-1})-3(2x^2-3x^{-1})\\
&=(-3x^2)(2x^2)+(-3x^2)(-3x^{-1})+(5x)(2x^2)\\
&\quad+(5x)(-3x^{-1})+(-3)(2x^2)+(-3)(-3x^{-1})\\
&=(-3)(2)x^{2+2}+(-3)(-3)x^{2-1}+(5)(2)x^{1+2}\\
&\quad+(5)(-3)x^{1-1}+(-3)(2)x^2+(-3)(-3)x^{-1}\\
&= -6x^4+9x^1+10x^3-15x^0-6x^2+9x^{-1}\\
&= -6x^4+9x+10x^3-15-6x^2+9x^{-1}\\
&= -6x^4+10x^3-6x^2+9x-15+9x^{-1}.
\end{aligned}
$$

(5)

$$
\begin{aligned}
(2x^3-5x^2+3x)(-4x^3+x^2-2x)& \\
&=2x^3(-4x^3+x^2-2x)-5x^2(-4x^3+x^2-2x)+\\
&\quad+3x(-4x^3+x^2-2x)\\
&=(2x^3)(-4x^3)+(2x^3)(x^2)+(2x^3)(-2x)\\
&\quad+(-5x^2)(-4x^3)+(-5x^2)(x^2)+(-5x^2)(-2x)\\
&\quad+(3x)(-4x^3)+(3x)(x^2)+(3x)(-2x)\\
&=(2)(-4)x^{3+3}+(2)(1)x^{3+2}+(2)(-2)x^{3+1}\\
&\quad+(-5)(-4)x^{2+3}+(-5)(1)x^{2+2}+(-5)(-2)x^{2+1}\\
&\quad+(3)(-4)x^{1+3}+(3)(1)x^{1+2}+(3)(-2)x^{1+1}\\
&= -8x^6+2x^5-4x^4+20x^5-5x^4+10x^3-12x^4+3x^3-6x^2\\
&= -8x^6+2x^5+20x^5-4x^4-5x^4-12x^4+10x^3+3x^3-6x^2\\
&= -8x^6+22x^5-21x^4+13x^3-6x^2.
\end{aligned}
$$

Remark 1.5.4. I recommend that you approach these problems systematically. For example, note how I have started with the expression on the left and taken each of its terms in order and multiplied them by the whole of the expression on the right. There is no mathematical reason why you have to do it this way, for example, you could start with the expression on the right and multiply each of its terms with the whole of the expression on the left. What is very important though is that you take the same approach each time. If you don't then it will be almost certain that you will make a mistake.

1.5.3. *Long Division of Numbers*.

Having had a look at multiplication, we will now examine division. Unfortunately when dealing with algebraic expressions, division is a lot more complicated than it was when dealing with numbers. On the one hand, given an algebraic expression, say $x^2 + x + 1$, then division by this expression is the same as multiplication by $\dfrac{1}{x^2 + x + 1}$, as was the case with numbers. However, the problem is that this doesn't get us anywhere since we don't know how to multiply a general algebraic expression by $\dfrac{1}{x^2 + x + 1}$.

The solution is to use long division. You will have used this in school for dividing numbers and we will adapt this technique for algebraic expressions. Before we do this however, we will revise the technique as used for numbers. When we divide one number into another using long division, then there are essentially two different approaches we can take. Firstly, we can keep going until we have the quotient to any required degree of accuracy. Another approach is sometimes used if we are dividing a natural number a by another natural number b. We can find natural numbers q and r with $0 \leqslant r < b$ such that $a = qb + r$. In this case q is called the *quotient* and r is called the *remainder*. Note that we can also write $a = qb + r$ as $\dfrac{a}{b} = q + \dfrac{r}{b}$. It is this approach that is generalised when dealing with algebraic expressions, so this is what we will revise. Here are some examples.

Example 1.5.5.

(1)
$$
\begin{array}{r@{\,}l}
 & 8270 \\
7 \big) \!\!\! & \overline{57894} \\
 & \underline{56000} \\
 & 1894 \\
 & \underline{1400} \\
 & 494 \\
 & \underline{490} \\
 & 4
\end{array}
$$

So in this case we have $\dfrac{57894}{7} = 8270 + \dfrac{4}{7}$.

That is the quotient is 8270 and the remainder is 4.

(2)
$$
\begin{array}{r@{\,}l}
 & 12845 \\
8 \big) \!\!\! & \overline{102760} \\
 & \underline{80000} \\
 & 22760 \\
 & \underline{16000} \\
 & 6760 \\
 & \underline{6400} \\
 & 360 \\
 & \underline{320} \\
 & 40 \\
 & \underline{40} \\
 & 0
\end{array}
$$

So $\dfrac{102760}{8} = 12845$ and the remainder is zero in this case.

(3)
$$\begin{array}{r} 4382 \\ 56\,\big)\,\overline{245393} \\ \underline{224000} \\ 21393 \\ \underline{16800} \\ 4593 \\ \underline{4480} \\ 113 \\ \underline{112} \\ 1 \end{array}$$

So $\dfrac{245393}{56} = 4382 + \dfrac{1}{56}$.

That is the quotient is 4382 and the remainder is 1.

(4)
$$\begin{array}{r} 83927 \\ 672\,\big)\,\overline{56399502} \\ \underline{53760000} \\ 2639502 \\ \underline{2016000} \\ 623502 \\ \underline{604800} \\ 18702 \\ \underline{13440} \\ 5262 \\ \underline{4704} \\ 558 \end{array}$$

So $\dfrac{56399502}{672} = 83927 + \dfrac{558}{672}$.

That is the quotient is 83927 and the remainder is 558.

1.5.4. *Polynomial Long Division*.

Now that we have had a look at numbers let us move on to algebraic expressions. We will restrict ourselves to looking at sums of terms of the form ax^b where a and b are integers with b non-negative (technically these sorts of expressions are called polynomials with integer coefficients).

The technique is very similar to that used with numbers, the only difference is that instead of obtaining digits at the top, we will obtain terms of the form ax^b, where b is a non-negative integer and a is a rational number. The easiest way to see what I mean will be to go through some examples.

Example 1.5.6.

(1)

$$
\begin{array}{rl}
 & x - 1 \\
\cline{2-2}
x+2 \big) & x^2 \quad + x + 1 \\
 & \underline{- x^2 - 2x} \\
 & \qquad - x + 1 \\
 & \qquad \underline{\;\; x + 2} \\
 & \qquad\qquad 3
\end{array}
$$

This tells us that $\dfrac{x^2+x+1}{x+2} = x - 1 + \dfrac{3}{x+2}$.

So the quotient is $x-1$ and the remainder is 3.

(2)

$$
\begin{array}{rl}
 & \qquad x^2 \qquad + 1 \\
\cline{2-2}
x+1 \big) & x^3 + x^2 + x + 1 \\
 & \underline{- x^3 - x^2} \\
 & \qquad\qquad x + 1 \\
 & \qquad\qquad \underline{- x - 1} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

This tells us that $\dfrac{x^3+x^2+x+1}{x+1} = x^2 + 1$.

So the quotient is x^2+1 and the remainder is 0.

Note that this example shows us that if we know one factor of an expression, then we can use division to find the other factor. For example, say we knew that $x+1$ was a factor of x^3+x^2+x+1, then using division, we see that the other factor is x^2+1.

(3)

$$
\begin{array}{rl}
 & \qquad\qquad x^2 \quad - 2x + 4. \\
\cline{2-2}
3x+1 \big) & 3x^3 - 5x^2 + 10x - 3 \\
 & \underline{- 3x^3 \quad - x^2} \\
 & \qquad - 6x^2 + 10x \\
 & \qquad \underline{\;\;\; 6x^2 \quad + 2x} \\
 & \qquad\qquad\qquad 12x - 3 \\
 & \qquad\qquad\qquad \underline{- 12x - 4} \\
 & \qquad\qquad\qquad\qquad - 7
\end{array}
$$

This tells us that $\dfrac{3x^3-5x^2+10x-3}{3x+1} = x^2 - 2x + 4 + \dfrac{-7}{3x+1}$.

So the quotient is x^2-2x+4 and the remainder is -7.

Note that when dealing with algebraic expressions, the remainder does not have to be positive.

(4)

$$\begin{array}{rllll}
 & x^2 & -\tfrac{3}{2}x & +\tfrac{5}{4}. & \\
\cline{2-5}
2x+1 \big) & 2x^3 - 2x^2 & +x & +2 & \\
 & -2x^3 - x^2 & & & \\
\cline{2-2}
 & -3x^2 & +x & & \\
 & 3x^2 & +\tfrac{3}{2}x & & \\
\cline{2-3}
 & & \tfrac{5}{2}x & +2 & \\
 & & -\tfrac{5}{2}x & -\tfrac{5}{4} & \\
\cline{3-4}
 & & & \tfrac{3}{4} &
\end{array}$$

This tells us that $\dfrac{2x^3 - 2x^2 + x + 2}{2x+1} = x^2 - \dfrac{3}{2}x + \dfrac{5}{4} + \dfrac{\frac{3}{4}}{2x+1}$.

So the quotient is $x^2 - \dfrac{3}{2}x + \dfrac{5}{4}$ and the remainder is $\dfrac{3}{4}$.

(5)

$$\begin{array}{rlllll}
 & & & 4x^2 & -x & -7. \\
\cline{2-6}
x^2+x+2 \big) & 4x^4 & +3x^3 & & +2x & +1 \\
 & -4x^4 & -4x^3 & -8x^2 & & \\
\cline{2-4}
 & & -x^3 & -8x^2 & +2x & \\
 & & x^3 & +x^2 & +2x & \\
\cline{3-5}
 & & & -7x^2 & +4x & +1 \\
 & & & 7x^2 & +7x & +14 \\
\cline{4-6}
 & & & & 11x & +15
\end{array}$$

This tells us that $\dfrac{4x^4 + 3x^3 + 2x + 1}{x^2 + x + 2} = 4x^2 - x - 7 + \dfrac{11x + 15}{x^2 + x + 2}$.

So the quotient is $4x^2 - x - 7$ and the remainder is $11x + 15$.

Note that the remainder does not even have to be a number but what is true is that the highest power of x in the remainder has to be lower than the highest power of x in the divisor.

Remark 1.5.7. As always in maths, it is good practice to check our answer once we have obtained it and this is especially important when there are plenty of places to go wrong, as in this case. Fortunately there is a good way of checking our answer here; we have to check that the final equation holds. For example, let us have a look at the last example. We have to check that

$$\frac{4x^4 + 3x^3 + 2x + 1}{x^2 + x + 2} = 4x^2 - x - 7 + \frac{11x + 15}{x^2 + x + 2}$$

or equivalently that $4x^4 + 3x^3 + 2x + 1 = (4x^2 - x - 7)(x^2 + x + 2) + 11x + 15$. So we just have to multiply out $(4x^2 - x - 7)(x^2 + x + 2) + 11x + 15$ and check that it equals $4x^4 + 3x^3 + 2x + 1$.

This does take quite a bit of time however, so in an exam it may not be appropriate to do this full check. However, even if we are short of time, we can at least make sure that the highest power terms on each side of the equals sign are equal, i.e., that $4x^4 = (4x^2)(x^2)$ in this case. Another useful check is that the highest power of x

in the remainder (one in this case) must be less than the highest power of x in the divisor (two in this case).

1.6. **The Binomial Theorem.**

We have already had a look at finding powers of algebraic expressions in Example 1.5.3(3) where we found the square of $2x^2 + 1$. While this method can be used to find higher powers, it is very complicated even for the third power and it is very easy to make mistakes. The Binomial Theorem gives us a much easier method, which is a lot less prone to mistakes.

I will start by stating the theorem and then explain what it all means.

Theorem 1.6.1 (The Binomial Theorem)**.** *Let x and y be algebraic expressions and let n be a natural number, then*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i. \tag{2}$$

1.6.1. *Summation Notation*.

There are two pieces of notation on the right hand side of (2) that you may not be familiar with, so let us look at them in turn.

The first is the *summation* sign $\sum_{i=0}^{n}$. The sign $\sum$ tells us that we have to add up a collection of expressions and the $i = 0$ and the n tell us exactly how to do this. The i is called a *dummy variable* since it can be changed to another letter without affecting the sum. For example $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ means exactly the same as (2); note we have replaced i with k on the right hand side. What the $k = 0$ and the n mean is that we start by putting $k = 0$ in $\binom{n}{k} x^{n-k} y^k$, then we put $k = 1$, then $k = 2$ and so on up until $k = n$ and then add all the terms up. Here are some examples of 'summation notation'.

Example 1.6.2.

(1) $\sum_{i=0}^{3} i = 0 + 1 + 2 + 3 = 6.$

(2) $\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$

(3) $\sum_{i=-1}^{2} i^3 = (-1)^3 + 0^3 + 1^3 + 2^3 = -1 + 0 + 1 + 8 = 8.$

(4) $\sum_{i=2}^{4} x^i = x^2 + x^3 + x^4.$

(5) $\sum_{i=1}^{4} ix^i = x + 2x^2 + 3x^3 + 4x^4.$

Remark 1.6.3. Note that summation notation can be used with numbers or with algebraic expressions.

This also means that (2) can be written as

$$
\begin{aligned}
(x+y)^n &= \binom{n}{0}x^{n-0}y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^{n-n}y^n \\
&= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n. \qquad (3)
\end{aligned}
$$

1.6.2. *Factorials and Binomial Coefficients*.

Next we have to have a look at $\binom{n}{i}$. This is called a *binomial coefficient* and you may have met the other notation ${}^nC_i$ in school. It tells you the number of ways you can choose i items from n items (where order doesn't matter) and the numbers $\binom{n}{i}$ appear in Pascal's triangle, which I have shown in Figure 3.

$$
\begin{array}{lccccccccccccccccccc}
n=0 & & & & & & & & & & 1 & & & & & & & & & \\
n=1 & & & & & & & & & 1 & & 1 & & & & & & & & \\
n=2 & & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
n=3 & & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
n=4 & & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\
n=5 & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & \\
n=6 & & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & \\
n=7 & & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & \\
n=8 & & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & \\
n=9 & 1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1
\end{array}
$$

FIGURE 3. Pascal's triangle.

To find $\binom{n}{i}$ we look at the appropriate row and go to the $i+1$'th number from the left. So, for example, $\binom{7}{0} = 1$ since the first number in the $n=7$ row is 1 and $\binom{7}{1} = 7$ since the second number in the $n=7$ row is 7.

There are also a couple of interesting features to notice. Firstly, the triangle is symmetric about the central vertical line, so we have $\binom{n}{i} = \binom{n}{n-i}$. Secondly, each number in the triangle is the sum of the two numbers in the line above immediately to the left and the right. This means that $\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}$. We will record these two features as a theorem.

Theorem 1.6.4 (Binomial coefficient identities)**.** *If n and i are non-negative integers, then*

(1) $\binom{n}{i} = \binom{n}{n-i}$.

(2) $\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}$.

While we can read off the value of $\binom{n}{i}$ from Pascal's triangle for small values of n, we need another method if n is large. Fortunately there is a formula that enables us to calculate binomial coefficients for any value of n. I will state it first and then explain the notation.

Theorem 1.6.5 (Calculation of binomial coefficients)**.** *If n and i are non-negative integers, then*

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}. \tag{4}$$

We now have to explain the notation $n!$

Definition 1.6.6 (Factorial)**.** If n is a natural number then

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

We also define $0! = 1$.

So, for example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. Note that the size of factorials rise extremely rapidly; even $60!$ is approximately 8.3×10^{81} and $70!$ is too large to be calculated by most standard calculators. Luckily this doesn't mean that we can't calculate $\binom{n}{i}$ when $n = 70$, say, since we can cancel a lot of the terms in (4) as I will now show.

$$\begin{aligned} \frac{n!}{i!(n-i)!} &= \frac{n(n-1)(n-2)\cdots(n-i+1)(n-i)(n-i-1)\cdots(2)(1)}{i!(n-i)(n-i-1)\cdots(2)(1)} \\ &= \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!}. \end{aligned}$$

Thus we also have the following.

Corollary 1.6.7 (Calculation of binomial coefficients)**.** *If n and i are natural numbers, then*

$$\binom{n}{i} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!}. \tag{5}$$

Remark 1.6.8. Corollary 1.6.7 does not work if either n or i is zero, since in these cases, (5) does not make any sense. In other cases it is usually (5) rather than (4) that we will use if we are calculating binomial coefficients by hand.

Before we do some examples, we also note that Theorem 1.6.4(1) is very useful if we want to calculate a binomial coefficient where i is close to n, since this reduces the number of terms we have to calculate substantially. For example, if we wanted to calculate $\binom{70}{69}$, then there would be 69 terms on the numerator and 68 terms on the denominator of (5) (most of which cancel). On the other hand, using Theorem 1.6.4(1), $\binom{70}{69} = \binom{70}{70-69} = \binom{70}{1}$ which then equals $\frac{70}{1} = 70$ using (5).

Here are some more examples of calculating binomial coefficients.

Example 1.6.9.

(1) $\binom{10}{1} = \frac{10}{1} = 10$.

(2) $\binom{23}{2} = \frac{23 \times 22}{2} = 253$.

(3) $\binom{8}{0} = \frac{8!}{0!8!} = \frac{8!}{(1)8!} = 1$.
Here we used (4) rather than (5).

(4) $\binom{102}{99} = \binom{102}{3} = \frac{102 \times 101 \times 100}{3 \times 2} = 171700$.
Here we first used Theorem 1.6.4(1) to make (5) easier to use.

1.6.3. *The Binomial Theorem*.

We can now do some examples on The Binomial Theorem itself but first we will note that since $\binom{n}{0} = \binom{n}{n} = 1$, (3) can be written as

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + y^n.$$

Example 1.6.10.

(1) $(x+y)^2 = x^2 + \binom{2}{1}xy + y^2 = x^2 + 2xy + y^2$.

(2) $(x+y)^3 = x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + y^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
Note that even for an example like this, a mistake is much more likely if we just multiply the brackets out.

(3)

$$\begin{aligned}(x+y)^4 &= x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

(4) $(1+y)^3 = 1^3 + \binom{3}{1}1^2y + \binom{3}{2}(1)y^2 + y^3 = 1 + 3y + 3y^2 + y^3$.

(5) $\left(x^2+y^3\right)^2=\left(x^2\right)^2+\binom{2}{1}\left(x^2\right)\left(y^3\right)+\left(y^3\right)^2=x^4+2x^2y^3+y^6.$

(6)

$$\begin{aligned}(2x-1)^4&=(2x)^4+\binom{4}{1}(2x)^3(-1)+\binom{4}{2}(2x)^2(-1)^2+\binom{4}{3}(2x)(-1)^3+(-1)^4\\&=16x^4+4\left(8x^3\right)(-1)+6\left(4x^2\right)(1)+4(2x)(-1)+1\\&=16x^4-32x^3+24x^2-8x+1.\end{aligned}$$

(7)

$$\begin{aligned}\left(3x-2y^{-2}\right)^4=&(3x)^4+\binom{4}{1}(3x)^3\left(-2y^{-2}\right)+\binom{4}{2}(3x)^2\left(-2y^{-2}\right)^2\\&+\binom{4}{3}(3x)\left(-2y^{-2}\right)^3+\left(-2y^{-2}\right)^4\\=&81x^4+4\left(27x^3\right)\left(-2y^{-2}\right)+6\left(9x^2\right)\left(4y^{-4}\right)+4(3x)\left(-8y^{-6}\right)+16y^{-8}\\=&81x^4-216x^3y^{-2}+216x^2y^{-4}-96xy^{-6}+16y^{-8}.\end{aligned}$$

(8)

$$\begin{aligned}(3-5)^3&=3^3+\binom{3}{1}3^2(-5)+\binom{3}{2}(3)(-5)^2+(-5)^3\\&=27+(3)(9)(-5)+3(3)(25)-125\\&=-8.\end{aligned}$$

Since $(3-5)^3=-8$, this example shows that we get what we expect when x and y are numbers.